Solution 13

Supplementary Problems

1. Verify the identity
$$\nabla \times \nabla f = \mathbf{0}$$
for any function f.

   Solution. This is a direct computation. The converse is interesting: If $\nabla \times \mathbf{F} = 0$ for some vector field $\mathbf{F}$, is there some function f such that $\mathbf{F} = \nabla f$? The answer is yes when $\mathbf{F}$ is defined in a simply-connected region. Note that the condition $\nabla \times \mathbf{F} = 0$ is nothing but the component test.

2. Verify the identity
$$\nabla \cdot \nabla \times \mathbf{F} = 0$$
for any vector field $\mathbf{F}$. Use this fact to show that $x\mathbf{i} + y\mathbf{j} + x^2z\mathbf{k}$ cannot be the curl of some vector field.

   Solution. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. We have
$$\nabla \cdot \nabla \times \mathbf{F} = (P_y - N_z)_x - (P_x - M_z)_y + (N_x - M_y)_z = 0 \ .$$
If there is some $\mathbf{F}$ such that $\nabla \times \mathbf{F} = x\mathbf{i} + y\mathbf{j} + x^2z\mathbf{k}$, then $\nabla \cdot \mathbf{F} = 0$, but now
$$\nabla \cdot (x\mathbf{i} + y\mathbf{j} + x^2z\mathbf{k}) = 2 + x^2 > 0,$$
contradiction holds.

   Note. Like in (a), we could raise the converse question: When $\nabla \cdot \mathbf{G} = 0$, can we find a vector field $\mathbf{F}$ so that $\nabla \times \mathbf{F} = \mathbf{G}$? The answer again relies on the topology of the region in which $\mathbf{G}$ defines. It is yes when the region is the entire space.